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On the Asymptotic Equivalence between the Enskog and the Boltzmann Equations

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An asymptotic equivalence theorem is proven between the solutions of the initial value problem in all space for the Boltzmann and Enskog equations for initial data which assure global existence for the solutions to the initial value problem for one of the two equations. The proof is given starting from the solution of the Boltzmann equation, then the proof line is simply indicated when one starts from the Enskog equation. The proof holds for Knudsen numbers of the order of unity and equivalence is proven when the scale of the dimensions of the gas particles characterizing the Enskog equation tends to zero.

KEY WORDS: Kinetic theory; Boltzmann equation; Enskog equation; hydrodynamics; asyptotic equivalence; initial value problem.

1. INTRODUCTION

The Boltzmann equation is a mathematical model, in the kinetic theory of gases[(10,13)] which describes the time-space evolution of the one-particle distribution function of a dilute monatomic gas. The interaction between particles is described by suitable pairwise interaction potentials for mass point particles.[(6,7)]

The Enskog equation[(5,13)] is a mathematical model for moderately dense gases such that each particle is modeled as a sphere with finite diameter with interaction between pairs of particles according to a hard-sphere potential. In addition, the overall dimensions of the spheres are taken into account in estimating the collision frequency. The Enskog

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equation in its standard form, classified in Ref. 2 by the abbreviation SEE, was proposed by Enskog[5] under semiempirical arguments discussed in detail in the sixth chapter of Ref. 13. This model is not consistent with irreversible thermodynamics. The original model has been modified[1] under more detailed derivation rules in order to provide an equation consistent with irreversible thermodynamics. This new model was classified in Ref. 2 with the abbreviation REE (revised Enskog equation).

This paper deals with the analysis of the asymptotic equivalence (when the scale of the dimensions of the gas particles in the Enskog model tends to zero) between the Boltzmann and the Enskog equations when the initial conditions are such that a global existence theorem holds for the solutions to the Cauchy problem for the Boltzmann equation. In particular, the analysis will deal with small initial conditions tending to zero at infinity in the phase space.

This analysis can be regarded as an indirect validation of the Enskog model and will include, as a particular result, the analysis of global existence and uniqueness of the solutions to the Cauchy problem for the Enskog equation. The mathematical results proposed in this paper hold for the SEE and can generally be extended to the REE. In this case, following the terminology proposed in Ref. 2 it will be stated that all results hold for the Enskog equation in general, i.e., for the EE.

The analysis starts from the fact that the initial value problem for the Boltzmann equation in all space, as well as for the Enskog equation,[15] has a unique global mild solution if the initial datum decays rapidly enough at infinity with respect both to space and velocity. This is true even if the total mass of the gas is infinite (for sufficiently smooth decay in the physical space), provided that the initial datum is small in norm.[3,14]

Section 2 provides a description of the Boltzmann and Enskog equations written in a classical dimensionless form. Then a slight generalization of the existence theorems given in Refs. 3 and 14 is provided in such a fashion that if the initial datum is smooth in space, then a certain smoothness characterizes the solution. Given this solution, it is proven in the last section that the Enskog equation for the same initial datum also has a global solution as long as the particle diameter is small and that this solution actually approaches that of the Boltzmann equation when the diameter tends to zero. In particular, an evolution equation is derived for the difference between the solutions of the same initial value problem for the two mathematical models. The main theorem of the paper proves global existence and boundness properties of such a remainder. In addition, it is proven that the difference tends to zero when the diameter of the particles tends to zero. The theorem can also be proven, under some technical extensions, starting from the Enskog equation and showing that the solution of the Boltzmann equations for the same initial datum exists globally in time and tends to that of the Enskog equation in the same asymptotic limit.

2. THE MATHEMATICAL MODEL

Both the Boltzmann and the Enskog equations describe the time-space evolution of the one-particle distribution function

$$\tilde{f} = \tilde{f}(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{V}}): \quad \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+ \tag{1}$$

the moments of which provide the macroscopic hydrodynamic quantities, such as the local mass density

$$\rho = \rho(\tilde{t}, \tilde{\mathbf{x}}) = mn(\tilde{t}, \tilde{\mathbf{x}}) = m \int_{\mathbb{R}^3} \tilde{f}(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{V}})\, d\tilde{\mathbf{V}} \tag{2}$$

mass velocity

$$\mathbf{U} = \mathbf{U}(\tilde{t}, \tilde{\mathbf{x}}) = [1/n(\tilde{t}, \tilde{\mathbf{x}})] \int_{\mathbb{R}^3} \tilde{\mathbf{V}} \tilde{f}(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{V}})\, d\tilde{\mathbf{V}} \tag{3}$$

and temperature

$$T = T(\tilde{t}, \tilde{\mathbf{x}}) = [1/(3k/mn(\tilde{t}, \tilde{\mathbf{x}})] \int_{\mathbb{R}^3} (\tilde{\mathbf{V}} - \mathbf{U})^2 f(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{V}})\, d\tilde{\mathbf{V}} \tag{4}$$

See the reviews in Refs. 4, 6, and 7 for a detailed analysis of the mathematical problems and results related to the Boltzmann equation and Ref. 2 for a review of analogous problems related to the Enskog equation, either SEE or REE.

Recall that in the Enskog model pairwise collisions occur between gas particles with center in $\tilde{\mathbf{x}}$ and velocity $\tilde{\mathbf{V}}$ with particles with center in $(\tilde{\mathbf{x}} - \tilde{\sigma}\mathbf{v})$ and velocity $\tilde{\mathbf{V}}_1$, where $\mathbf{v}$ is the unit vector joining the centers of two hard spheres with diameter σ. The velocities after the collision can be classically recovered by the conservation equations of momentum and energy and will be indicated by $\tilde{\mathbf{V}}'$ and $\tilde{\mathbf{V}}'_1$. The collision, in the Boltzmann model, occurs in $\tilde{\mathbf{x}}$, i.e., a mass-point model is adopted for the gas particles. In addition, the collision frequency is increased in the Enskog model by a factor χ, which takes into account the overall dimensions of the gas particles, whereas this factor is equal to one in the Boltzmann model.

The initial value problem in all space with initial conditions $\tilde{F} = \tilde{f}(\tilde{t} = 0, \tilde{\mathbf{x}}, \tilde{\mathbf{V}})$ is considered in this paper for both equations written in the dimensionless form suggested in Section 11 of Chapter 2 in Ref. 10.

Such a dimensionless form is obtained by selecting some reference values for the independent variables $(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{V}})$ and for the distribution function $\tilde{f}$ and referring all dimensional quantities to the reference values. The value of the distribution function $\tilde{f}(\tilde{t}=0, \tilde{\mathbf{x}}=0, \tilde{\mathbf{V}})$ can be used to recover the reference quantities as indicated in Ref. 10, pp. 99–104. In particular, the characteristic time t_c is the mean collision time, the characteristic velocity V_c the mean velocity of the gas particles, the characteristic distribution function $f_c = n_0/c_0^3$, where n_0 and c_0 are, respectively, the number density and the thermal velocity in $(\tilde{\mathbf{x}}=0, \tilde{t}=0)$, and finally the characteristic length l_c can be chosen in order to have the Strouhal number $\mathrm{Sh} = l_c/V_c t_c$ equal to one.

Following the line indicated above, the two equations in the new dimensionless variables, which will simply be denoted without the tilde, i.e., $f = f(t, \mathbf{x}, \mathbf{V})$, take the form

$$\partial f_{\mathrm{B}}/\partial t + \mathbf{V} \cdot \nabla f_{\mathrm{B}} = (1/\mathrm{Kn})\, J_0(f_{\mathrm{B}}, f_{\mathrm{B}}) \tag{5}$$

$$\partial f_{\mathrm{E}}/\partial t + \mathbf{V} \cdot \nabla f_{\mathrm{E}} = (1/\mathrm{Kn})\, E_\sigma((1/\mathrm{Kn}) \cdot f_{\mathrm{E}}; f_{\mathrm{E}}, f_{\mathrm{E}}) \tag{6}$$

with initial conditions

$$F = f_{\mathrm{B}}(t=0, \mathbf{x}, \mathbf{V}) = f_{\mathrm{E}}(t=0, \mathbf{x}, \mathbf{V}) \tag{7}$$

In Eqs. (5) and (6), Kn is the Knudsen number defined by the ratio between the mean free path and the characteristic length in $t=0$ and $\mathbf{x}=0$. The collision operators for a hard-sphere gas with dimensionless diameter σ have been indicated by J_0 and E_σ. Consider now the following operators:

$$J_\sigma^+(f, g)(\mathbf{x}, \mathbf{V}) = \int_{\mathbb{R}^3 \times S^2} f(\tilde{\mathbf{x}} + \sigma\mathbf{v}, \mathbf{V}_1')\, g(\mathbf{x}, \mathbf{V}') \times \Phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1 \tag{8}$$

$$R_\sigma(f)(\mathbf{x}, \mathbf{V}) = \int_{\mathbb{R}^3 \times S^2} f(\tilde{\mathbf{x}} - \sigma\mathbf{v}, \mathbf{V}_1) \times \Phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1 \tag{9}$$

and

$$E_\sigma^+(f; g, h)(\mathbf{x}, \mathbf{V}) = \int_{\mathbb{R}^3 \times S^2} \chi^+(f; \sigma)\, g(\mathbf{x} + \sigma\mathbf{v}, \mathbf{V}_1')\, h(\mathbf{x}, \mathbf{V}') \times \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1 \tag{10}$$

$$E_\sigma^-(f; g, h)(\mathbf{x}, \mathbf{V}) = h(\mathbf{x}, \mathbf{V}) \int_{\mathbb{R}^3 \times S^2} \chi^-(f; \sigma)\, g(\mathbf{x} - \sigma\mathbf{v}, \mathbf{V}_1) \times \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1 \tag{11}$$

where

$$S^2 = \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| = 1\}, \qquad y \geqslant 0: \Phi(y) = y, \qquad y < 0: \Phi(y) = 0$$

χ^+ is the pair correlation function.

In addition, the following operators can be further defined as

$$J_\sigma(f, g) = \tfrac{1}{2}[J_\sigma^+(f, g) + J_\sigma^+(g, f) - fR_\sigma(g) - gR_\sigma(f)] \tag{12}$$

$$E_\sigma(f; g, h) = \tfrac{1}{2}[E_\sigma^+(f; g, h) + E_\sigma^+(f; h, g) - E_\sigma^-(f; g, h) - E_\sigma^-(f; h, g)] \tag{13}$$

Here J_0 is the classical Boltzmann operator for a hard-sphere interaction potential,[4,6,7] J_σ is the Boltzmann–Enskog operator, and E_σ is the Enskog operator.[2] Note that J_0 is a bilinear symmetric operator acting only upon the variable $\mathbf{V}$. Equations (5), (6) have now been fully characterized.

If one now considers the expansion of the operator J_σ in terms of powers of σ, the various terms of the expansion can be written as

$$J_\sigma^{(i)}(f, g) = \tfrac{1}{2}[J_\sigma^{+(i)}(f, g) + J_\sigma^{+(i)}(g, f) - fR_\sigma^{(i)}(g) - gR_\sigma^{(i)}(f)] \tag{14}$$

where

$$J_\sigma^{+(i)}(f, g) = \int_{\mathbb{R}^3 \times S^2} (\mathbf{v} \cdot \nabla)^i f(\mathbf{x} + \sigma\mathbf{v}, \mathbf{V}_1') \, g(\mathbf{x}, \mathbf{V}') \, \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v}) \, d\mathbf{v} \, d\mathbf{V}_1 \tag{15}$$

and

$$R_\sigma^{(i)}(f) = \int_{\mathbb{R}^3 \times S^2} (\mathbf{v} \cdot \nabla)^i f(\mathbf{x} - \sigma\mathbf{v}, \mathbf{V}_1) \, \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v}) \, d\mathbf{v} \, d\mathbf{V}_1 \tag{16}$$

Moreover, the expression for the remainder in the power expansion is

$$\hat{J}_\sigma^{(k)}(f, g) = (1/\sigma^k) \left[J_\sigma(f, g) - J_0(f, g) - \sum_{i=1}^{k-1} \sigma^i J_0^{(i)}(f, g) \right] \tag{17}$$

This paper deals with the analysis of the asymptotic equivalence between Eqs. (5) and (6) for the solutions of the initial value problem in all space with initial conditions (7). This analysis is developed in the next two sections. In particular, Section 3 deals with the analysis of the existence problem for the solutions to the Cauchy problem for the Boltzmann equation (5), whereas the Section 4 deals with the analysis of the asymptotic equivalence between the solutions of the two equations for initial conditions such that global existence is assured for the Boltzmann equation.

3. GLOBAL EXISTENCE RESULTS FOR THE BOLTZMANN EQUATION

Global existence and uniqueness results of the solutions to the Cauchy problem for the nonlinear Boltzmann equation in all space have been given, for sufficiently small initial data decaying to zero at infinity in the phase space, by Illner and Shinbrot,[9], Hamdache,[8] Bellomo and Toscani,[3] and Toscani.[14] All the above results have been reviewed and discussed in detail in Ref. 4. In particular, Toscani's global existence theorem[14] assumes the weakest decay in the phase space and will be the starting point of the analysis developed in this section, where a global existence theorem is proven in a suitable space of differentiable functions, whereas the original result was given simply for continuous and bounded functions. This "technical" extension also holds in the case of the global existence theorems proposed in Refs. 3, 8, and 9.

Consider, then, the function f along the free-stream trajectories

$$f^{\#}(t, \mathbf{x}, \mathbf{V}) = f(t, \mathbf{x} + \mathbf{V}t, \mathbf{V}) \tag{18}$$

and problems (5) and (6) in the "mild" form

$$f_{\mathrm{B}}^{\#} = F + (1/\mathrm{Kn})(\mathscr{U} J_0(f_{\mathrm{B}}, f_{\mathrm{B}}))^{\#} \tag{19a}$$

$$f_{\mathrm{E}}^{\#} = F + (1/\mathrm{Kn})(\mathscr{U} E_\sigma((1/\mathrm{Kn}) \cdot f_{\mathrm{E}}; f_{\mathrm{E}}, f_{\mathrm{E}}))^{\#} \tag{19b}$$

where the operator $\mathscr{U}$ is defined by

$$\mathscr{U} f(t, \mathbf{x}, \mathbf{V}) = \int_0^t f(s, \mathbf{x} - (t-s)\,\mathbf{V}, \mathbf{V})\, ds \tag{20}$$

Now let W_α be a function with values $W_\alpha(y) = (1 + y^2)^{\alpha/2}$ and $C_b^n(X)$ the space of all real functions continuous and bounded with all their derivatives of order $|\gamma| \leqslant n$, on the space X. Then the space

$$\mathbb{B}^n = \left\{ f \in C_b^0(\mathbb{R}^3 \times \mathbb{R}^3) \colon \left(\frac{\partial^{|\gamma|}}{\partial x^\gamma} f(\mathbf{x}, \mathbf{V}) \right) W_p(|\mathbf{x}|)\, W_k(|\mathbf{V}|) \in C_b^0(\mathbb{R}^3 \times \mathbb{R}^3) \text{ for each multiindex } \gamma \text{ such that } |\gamma| \leqslant n \right\} \tag{21}$$

can be defined and endowed with the norm

$$\|f\|_n = \sup_{|\gamma| \leqslant n,\ (\mathbf{x}, \mathbf{V}) \in \mathbb{R}^6} \left[\frac{\partial^{|\gamma|}}{\partial x^\gamma} f(\mathbf{x}, \mathbf{V})\, W_p(|\mathbf{x}|)\, W_k(|\mathbf{V}|) \right] \tag{22}$$

In addition, the space

$$\mathscr{B}^n = \Bigg\{ f \in C_b^0(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3): \left(\frac{\partial^{|\gamma|}}{\partial x^\gamma} f\right)^{\#} (t, \mathbf{x}, \mathbf{V})\, W_p(|\mathbf{x}|)\, W_k(|\mathbf{V}|) \in C_b^0(\mathbb{R}_+ + \mathbb{R}^3 \times \mathbb{R}^3) \text{ for each multiindex } \gamma \text{ such that } |\gamma| \leqslant n \Bigg\} \tag{23}$$

can also be defined with the norm

$$|||f|||_n = \sup_{t \in \mathbb{R}_+} \|f^{\#}(t)\|_n \tag{24}$$

It will be assumed in what follows that p and k are fixed constants and $p > 1$, $(k < 3)$; see Refs. 14 and 4. Toscani's Theorem[14] (see also Ref. 4) can be rewritten as follows:

Theorem 1. Let $f, g \in \mathscr{B}^0$; then the following inequality holds:

$$|||(\mathscr{U}J_0(f, g)^{\#}|||_0 \leqslant c_1\, |||f|||_0\, |||g|||_0 \tag{25}$$

where $c_1 = c/\mathrm{Kn}$ and c depends only on the values of p and k. In addition, there exists a critical value $c_2 = 1/(4c_1)$ such that if $F \geqslant 0$ and $\|F\|_0 < c_2$, then problem (19a) has a unique global nonnegative solution $f_{\mathrm{B}} \in \mathscr{B}^0$ and

$$|||f_{\mathrm{B}}|||_0 \leqslant 2\, \|F\|_0 \tag{26}$$

Remark 1. Note that inequality (26) implies

$$|||f_B|||_0 < 1/(2c_1) \tag{27}$$

Remark 2. The proof provided in Ref. 14 as well as in Ref. 4 was realized for the operator J_0^+; however, the same proof holds for the full operator J_0.

Remark 3. Considering that we deal with Knudsen numbers larger than zero, we will put $\mathrm{Kn} = 1$ for simplicity of writing in what follows.

Consider now the problem with smooth initial data in the function space defined in (23); a technical extension of Theorem 1 is the following

Theorem 2. Let the initial conditions be such that the smallness conditions defined in Theorem 1 are satisfied and in addition $F \in \mathbb{B}^n$; then problem (19a) has a unique global solution $0 \leqslant f_{\mathrm{B}} \in \mathscr{B}^n$.

Proof. The statement of Theorem 2 is such that the global existence conditions required by Theorem 1 are satisfied. Then there exists a unique solution f_B such that inequality (27) is satisfied. Consider now, for Kn = 1, in the terms defined by Remark 3, the evolution equation for $g = \partial f_B/\partial x_i$ in the form

$$g^{\#} = G + 2(\mathscr{U} J_0(f_B, g))^{\#} \tag{28}$$

where $G = \partial F/\partial x_i$. Equation (28), which is linear in g, can be solved by application of a fixed-point theorem, taking into account inequalities (25) and (27). One obtains

$$|||g|||_0 \leqslant \|G\|_0/(1 - 2c_1\, |||f_B|||_0) \tag{29}$$

and the statement of Theorem 2 follows by induction. ▮

4. GLOBAL EXISTENCE FOR THE ENSKOG EQUATION AND ASYMPTOTIC EQUIVALENCE

The problem of the analysis of the global existence of the solutions to the Cauchy problem for the Enskog equation and of the asymptotic equivalence with the Boltzmann equation is considered here. Before entering into details of the analysis of this problem, it is important to define the relevant mathematical properties of the term χ, which is a correction term of the collision frequency due to the overall dimensions of the gas particles. As already pointed out,[2,15] this factor is a monotonically increasing functional of the local density and becomes infinite when the local density approaches a critical value corresponding to the condensation density. Then only moderate densities can be taken into account. The set of these functions can be indicated by $\mathbb{D}_\sigma$ and can be defined as

$$\mathbb{D}_\sigma = \left\{ f\colon \sup_{t,\mathbf{x}} \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{V})\, d\mathbf{V} \leqslant K_\sigma \right\} \tag{30}$$

where K_σ is a given constant such that $\sigma \to 0 \Rightarrow K_\sigma \to \infty$.

Remark 4. The structure of the function space $\mathscr{B}^n$ is such that there exists a constant $c_3(K_\sigma)$ such that the following implication holds:

$$|||f|||_0 \leqslant c_3(K_\sigma) \Rightarrow f \in \mathbb{D}_\sigma \tag{31}$$

Then, independently of σ, one can choose a value K^* larger than $1/(2c_1)$ and find a critical value σ^* of σ so that $K^* \leqslant c_3(K_{\sigma^*})$ defines the subset

$$\mathbb{D}^* = \{ f\colon |||f|||_0 \leqslant K^* \} \tag{32}$$

Keeping this in mind and letting

$$\chi^{\pm}(f;\sigma)=1+\sigma\bar{\chi}_\sigma^{\pm}(f) \tag{33}$$

we can define the properties of the term χ by the following:

Hypotheses χ:

(i) $\chi^{+}(0)=0$.

(ii) $\forall f\in\mathbb{D}^*$: $-1\leqslant\bar{\chi}_\sigma^{\pm}(f)\leqslant\chi^*$, $\chi^*=O(1)$ for $\sigma\downarrow 0$.

(iii) $\forall f, g\in\mathbb{D}^*$: $f\leqslant g\Rightarrow\bar{\chi}_\sigma^{\pm}(f)\leqslant\bar{\chi}_\sigma^{\pm}(g)$.

(iv) $\forall f, g\in\mathbb{D}^*$: $|\chi_\sigma^{+}(f)-\chi_\sigma^{+}(g)|\leqslant l(\sigma)\,|\int_{\mathbb{R}^3}(f-g)\,d\mathbf{V}|$, where $l=O(1)$ for $\sigma\downarrow 0$.

In addition, we define $\bar{E}$ by the relation

$$E_\sigma(f;g,h)=J_\sigma(g,h)+\sigma\cdot\bar{E}_\sigma(f;g,h) \tag{34}$$

so that problem (19b) can be rewritten in the form

$$f_{\mathrm{E}}^{\#}=F+(\mathscr{U}J_\sigma(f_{\mathrm{E}},f_{\mathrm{E}}))^{\#}+\sigma(\mathscr{U}E_\sigma(f_{\mathrm{E}};f_{\mathrm{E}},f_{\mathrm{E}}))^{\#} \tag{35}$$

An approach similar to that of Ref. 12 can now be applied to derive an evolution equation for the difference between the solutions of the two equations. Then the solution to the Enskog equation is written in the form

$$f_{\mathrm{E}}(t,\mathbf{x},\mathbf{V})=f_{\mathrm{B}}(t,\mathbf{x},\mathbf{V})+\sigma\cdot r(t,\mathbf{x},\mathbf{V}) \tag{36}$$

where f_{B} is the solution of the Boltzmann equation given by Theorem 2. Problem (35) is then equivalent to the following evolution equation for the remainder:

$$\begin{aligned} r^{\#}&=2(\mathscr{U}J_\sigma(f_{\mathrm{B}},r))^{\#}+\sigma(\mathscr{U}J_\sigma(r,r))^{\#}+(\mathscr{U}\bar{E}_\sigma(f_{\mathrm{B}}+\sigma r;f_{\mathrm{B}},f_{\mathrm{B}}))^{\#}\\ &\quad+2\sigma(\mathscr{U}\bar{E}_\sigma(f_{\mathrm{B}}+\sigma r;f_{\mathrm{B}},r))^{\#}+\sigma^2(\mathscr{U}\bar{E}_\sigma(f_{\mathrm{B}}+\sigma r;r,r))^{\#}\\ &\quad+(\mathscr{U}\hat{J}_\sigma^{(1)}(f_{\mathrm{B}},f_{\mathrm{B}})) \end{aligned} \tag{37}$$

The following preliminary lemmas (the proofs of Lemmas 2 and 3 are given in the Appendix) are the first steps toward the mathematical analysis of problem (37).

Lemma 1. $\forall\mathbf{x},\mathbf{a}\in\mathbb{R}^3$ the following inequality holds:

$$W_\alpha(|\mathbf{x}|)/W_\alpha(|\mathbf{x}+\mathbf{a}|)\leqslant(1+|\mathbf{a}|+|\mathbf{a}|^2)^{\alpha/2} \tag{38}$$

Lemma 2. Let c_1 be the constant defined in Theorem 1, Eq. (25).

Then there exists a constant c_4 of the order of unity as $\sigma \downarrow 0$ such that the following inequality holds:

$$|||\mathscr{U} J_\sigma^{(1)}(f, f)|||_0 \leqslant (c_1 + \sigma c_4)\ |||f|||_1^2 \tag{39}$$

Remark 5. In the same fashion the following further inequality can be proven:

$$|||\mathscr{U} J_\sigma(f, g)|||_0 \leqslant (c_1 + \sigma c_4)\ |||f|||_0\ |||g|||_0 \tag{40}$$

Lemma 3. If $0 < \sigma < \sigma^*$, $|||f|||_0 \leqslant K^*$, and $g, h \in \mathscr{B}^0$, then

$$|||\mathscr{U} \bar{E}_\sigma(f; g, h)|||_0 \leqslant \chi^*(c_1 + \sigma c_4)\ |||g|||_0\ |||h|||_0 \tag{41}$$

and

$$\begin{aligned} &|||\mathscr{U} \bar{E}_\sigma(f_1; g_1, h_1) - \mathscr{U} \bar{E}_\sigma(f_2; g_2, h_2)|||_0 \\ &\quad \leqslant (c_1 + \sigma c_4)(c_5\ |||g_1|||_0\ |||h_1|||_0\ |||f_1 - f_2|||_0 \\ &\qquad + \chi^*\ |||h_1|||_0\ |||g_1 - g_2|||_0 + \chi^*\ |||g_2|||_0\ |||h_1 - h_2|||_0) \end{aligned} \tag{42}$$

where

$$c_5 = l\ \|W_{-k}\|_{L_1(\mathbb{R}^3)}$$

Remark 6. Remark 5 and Lemma 3 imply that the full Enskog operator E_σ satisfies the inequalities

$$|||\mathscr{U} E_\sigma(f; g, h)|||_0 \leqslant (c_1 + \sigma c_6)\ |||g|||_0\ |||h|||_0 \tag{43}$$

and

$$\begin{aligned} &|||\mathscr{U} E_\sigma(f_1; g_1, h_1) - \mathscr{U} E_\sigma(f_2; g_2, h_2)|||_0 \\ &\quad \leqslant \sigma c_7\ |||g_1|||_0\ |||h_1|||_0\ |||f_1 - f_2|||_0 \\ &\qquad + (c_1 + \sigma c_6)\ |||h_1|||_0\ |||g_1 - g_2|||_0 + (c_1 + \sigma c_6)\ |||g_2|||_0\ |||h_1 - h_2|||_0 \end{aligned} \tag{44}$$

where

$$c_6 = c_4 + c_1 x^* + \sigma^* c_4 x^*, \qquad c_7 = (c_1 + \sigma^* c_4)\ l\ \|W_{-k}\|_{L_1(\mathbb{R}^3)}$$

The main result of this paper, i.e., the theorem defining the asymptotic equivalence of the solutions to the Cauchy problem for the Boltzmann and the Enskog equations, can now be formulated.

Theorem 3. Let f_B be the mild solution of the Boltzmann equation given by Theorem 2 for $n = 1$, and K^* and σ^* be the critical values introduced in Remark 4; then there exists a constant σ_B such that if $0<\sigma\leqslant\sigma_B$ ($\leqslant\sigma^*$), then a unique mild solution f_E of the Enskog equation exists in $\mathscr{B}^0$. In addition,

$$|||f_E - f_B|||_0 \leqslant \text{const}\cdot\sigma \tag{45}$$

Proof. Consider the following constants:

$$\hat{a} = 2\ |||f_B|||_0\ [(c_1+\sigma^* c_4)\,x^* + c_4]$$

$$b = (1 + x^*\sigma^*)(c_1 + \sigma^* c_4)$$

$$c = (1 + x^*)(c_1 + \sigma^* c_4)$$

Following inequality (27), three values σ_1, σ_2, and σ_3, $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, can be chosen such that

$$a = \sigma_1 \hat{a} + 2c_1\ |||f_B|||_0 < 1 \tag{46a}$$

$$2\sigma_2 \frac{c}{1-a}\ |||f_B|||_1^2 \leqslant \frac{1}{c_1} \tag{46b}$$

$$\sigma_3 \frac{4bc}{(1-a)^2}\ |||f_B|||_1^2 \leqslant 1 \tag{46c}$$

In addition, let $\hat{\mathscr{B}}^0$ be a closed convex subset of the space $\mathscr{B}^0$ defined by the closure

$$\hat{\mathscr{B}}^0 = \left\{ r\in\mathscr{B}^0 \colon |||r|||_0 \leqslant \frac{2c}{1-a}\ |||f_B|||_1^2 \right\}$$

and $A_\sigma^\#$ be the operator on the right side of the evolution equation (37). Assume that $0<\sigma\leqslant\sigma_4=\min(\sigma_3,\sigma^*)$ and $r\in\hat{\mathscr{B}}^0$; then

$$f_B + \sigma r \in \mathbb{D}^*$$

and according to Lemmas 2 and 3 and Remark 5, one has

$$|||A_\sigma r|||_0 \leqslant a\ |||r|||_0 + \sigma b\ |||r|||_0^2 + c\ |||f_B|||_1^2 \tag{47}$$

Inequalities (46), (47) imply that if $r\in\hat{\mathscr{B}}^0$, then $A_\sigma r\in\mathscr{B}^0$. Finally, one can apply the inequalities of Lemma 3 and Remark 5 and choose $\sigma_B \leqslant \sigma_4$ so that if $0\leqslant\sigma\leqslant\sigma_B$, one obtains for $\eta<1$

$$|||A_\sigma r_1 - A_\sigma r_2|||_0 \leqslant \eta\ |||r_1 - r_2|||_0 \tag{48}$$

Then inequalities (47), (48) imply that A_σ is a contraction operator in the closed convex subset $\hat{\mathscr{B}}^0$; consequently, the evolution equation for the remainder has, according to the fixed-point theorem, a unique solution $r \in \hat{\mathscr{B}}^0$. In addition, the Enskog equation has a unique mild solution $f_E = (f_B + \sigma r) \in \hat{\mathscr{B}}^0$ and inequality (45) of Theorem 3 is satisfied. ∎

Remark 7. Inequality (45) implies that the distance between f_E and f_B tends to zero when σ also tends to zero. Theorem 3 does not provide a proof of the positivity of f_E, which can be obtained, in the same fashion as in Ref. 15, by applying the Kaniel and Shinbrot monotone iterative scheme to the evolution equation of the Enskog model.

One can deal with the reverse problem in the same fashion as in the above proof. Namely, one can start from an existence theorem for the Enskog equation and prove that under the same existence hypotheses one can obtain global existence for the Boltzmann equation as well as asymptotic equivalence. The starting point is now the main theorem of Ref. 15, which can be rewritten, after a proof analogous to that of Theorem 2, as follows:

Theorem 4. Let the initial conditions for the Enskog equation be given in $\mathbb{B}^n$, $F \in \mathbb{B}^n$, then there exists a critical constant (independent on σ) such that if $F \geqslant 0$ and $\|F\|_0 < c_E$, the initial value problem (19a) has a unique global solution $0 \leqslant f_E \in \mathscr{B}^n$.

Then, after calculations analogous to those developed in this section, one can prove the following result:

Theorem 5. Let f_E be the mild solution of the Enskog equation defined in Theorem 4 for $n = 1$; then, under a suitable smallness assumption on σ, a unique solution f_B exists to problem (19a) in $\mathscr{B}^0$ and, in addition,

$$\sigma \downarrow 0 \Rightarrow |||f_B - f_E|||_0 \to 0 \tag{49}$$

The proofs of Theorems 4 and 5 are technically the same as the proof that produced the main result contained in Theorem 3. It only needs slightly more detailed assumptions on the χ-functionals. Therefore there is no need to report these additional technical details. On the other hand, it is worth pointing out that the analysis developed in this paper can be regarded as a useful step toward the undertanding of the hydrodynamic limit for the Enskog equation, which is still an open problem.

APPENDIX

Proof of Lemma 2: If $f \in \mathscr{B}^1$, then one has

$$J_\sigma^{(1)}(f, f)(t, \mathbf{x}, \mathbf{V})$$
$$= \int_{\mathbb{R}^3 \times S^2} (\mathbf{v} \cdot \nabla f)(t, \mathbf{x} + \vartheta^+ \sigma \mathbf{v} + \mathbf{V}_1')\, f(t, \mathbf{x}, \mathbf{V}')\, \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1$$
$$- f(t, \mathbf{x}, \mathbf{V}) \int_{\mathbb{R}^3 \times S^2} (\mathbf{v} \cdot \nabla f)(t, \mathbf{x} - \vartheta^- \sigma \mathbf{v}, \mathbf{V}_1)$$
$$\times \phi((\mathbf{v}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1$$

where

$$\vartheta^+ = \vartheta^+(t, \mathbf{x}, \mathbf{V}_1'), \qquad \vartheta^- = \vartheta^-(t, \mathbf{x}, \mathbf{V}'), \qquad \vartheta^\pm \in \,]0, 1[$$

The integration of the above quantity over t has to be considered carefully because of the dependence of the term ϑ^+ on t. Consider first the "gain" component $\hat{J}_\sigma^{+(1)}$ of the operator $\hat{J}_\sigma^{(1)}$; denoting $g = \mathbf{v} \cdot \nabla f$, one has

$$|(\mathscr{U}\hat{J}_\sigma^+(f, f))^\#|$$
$$\leqslant \int_0^t \int_{\mathbb{R}^3 + S^2} |g(s, \mathbf{x} + s\mathbf{V} + \vartheta^+ \sigma \mathbf{v}, \mathbf{V}_1')| \cdot |f(s, \mathbf{x} + s\mathbf{V}, \mathbf{V}')|$$
$$\times \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1\, ds$$
$$= \int_0^t \int_{\mathbb{R}^3 \times S^2} |g^\#(s, \mathbf{x} + s(\mathbf{V} - \mathbf{V}_1') + \vartheta^+ \sigma \mathbf{v}, \mathbf{V}_1')|$$
$$|f^\#(s, \mathbf{x} + s(\mathbf{V} - \mathbf{V}'))|\, \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1\, ds$$
$$\leqslant |||g|||_0\, |||f_0||| \int_{\mathbb{R}^3 \cdot S^2} \int_0^t (W_{-p}(|\mathbf{x} + s(\mathbf{V} - \mathbf{V}_1') + \vartheta^+ \sigma \mathbf{v}|)$$
$$\times W_p(|\mathbf{x} + s(\mathbf{V} - \mathbf{V}_1')|)$$
$$\times W_{-p}(|\mathbf{x} + s(\mathbf{V} - \mathbf{V}_1')|)\, W_{-p}(|\mathbf{x} + s(\mathbf{V} - \mathbf{V}')|)\, ds$$
$$\times W_{-k}(|\mathbf{V}_1|)\, W_{-k}(|\mathbf{V}|)\, \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1$$

The application of Lemma 1 provides the following inequality:

$$|(\mathscr{U}\hat{J}_\sigma^{+(1)}(f, f))^\#|$$
$$\leqslant (1 + \sigma + \sigma^2)^{p/2}\, |||g|||_0\, |||f|||_0$$
$$\times \int_{\mathbb{R}^3 \times S^2} \int_0^t W_{-p}(|\mathbf{x} + s(\mathbf{V} - \mathbf{V}_1')|)\, W_{-p}(|\mathbf{x} + s(\mathbf{V} - \mathbf{V}')|)\, ds$$
$$\times W_{-k}(|\mathbf{V}|)\, W_{-k}(|\mathbf{V}_1|)\, \phi((\mathbf{V}_1 - \mathbf{V}) \cdot \mathbf{v})\, d\mathbf{v}\, d\mathbf{V}_1$$

Analogous treatment can be applied to the "loss" term in the operator $\hat{J}_\sigma^{(1)}$. Finally, the statement of Lemma 2 is obtained by application of Lemma 2.3 of Ref. 14; see also Theorem 2.1 of Ref. 4.

Proof of Lemma 3. The first inequality in Lemma 3 follows from Hypothesis χ (ii) on the χ function and from Remark 5. Consider now, in order to prove the second inequality, the following:

$$\begin{aligned} &|||\mathscr{U}\bar{E}_\sigma(f_1; g_1, h_1) - \mathscr{U}\bar{E}_\sigma(f_2; g_2, h_2)|||_0 \\ &\quad \leqslant |||\mathscr{U}\bar{E}_\sigma(f_1; g_1, h_1) - \mathscr{U}\bar{E}_\sigma(f_2; g_1, h_1)|||_0 + |||\mathscr{U}\bar{E}_\sigma(f_2; g_1 - g_2, h_1|||_0 \\ &\qquad + |||\mathscr{U}\bar{E}_\sigma(f_2; g_2, h_1 - h_2)|||_0 \end{aligned}$$

Hypothesis (iv) on the χ-function gives the following estimate on the first term on the right side of the above inequality:

$$l\,|||\mathscr{U}J_\sigma(g_1, h_1)|||_0 \int_{\mathbb{R}^2} \sup_{t,\mathbf{x}}((f_1 - f_2)(t, \mathbf{x} + t\mathbf{V}, \mathbf{V})\, W_p(|\mathbf{x}|)\, W_k(|\mathbf{V}|))\, W_{-k}(|\mathbf{V}|)\, d\mathbf{V}$$

Then the lemma is proven by straightforward calculations.

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REFERENCES

1. H. Van Beijeren and M. H. Ernst, *Physica* **68**:437 (1973).
2. N. Bellomo and M. Lachowicz, *Int. J. Mod. Phys. B* **1**:1193 (1987).
3. N. Bellomo and G. Toscani, *J. Math. Phys.* **1**:(1985), 334–338.
4. N. Bellomo and G. Toscani, *Int. Rep. Dip. Matematico Politecnico Torino, Italy*, No. 16 (1986).
5. D. Enskog, in *Kinetic Theory*, Vol. 3, S. Brush, ed. (Pergamon Press, New York, 1972).
6. W. Fiszdon, M. Lachowicz, and A. Palczewski, in *Trends and Application of Pure Mathematics to Mechanics*, P. G. Chiarlet and M. Roseau, eds. (Springer, Heidelberg, 1984), pp. 63–95.
7. W. Greenberg, J. Polewczak, and P. Zweifel, in *Nonequilibrium Phenomena: The Boltzmann Equation*, J. Lebowitz and E. Montroll, eds. (North-Holland, Amsterdam, 1983), pp. 21–49.
8. K. Hamdache, *Jpn. J. Appl. Math.* **2**:1 (1985).
9. R. Illner and M. Shinbrot, *Commun. Math. Phys.* **95**:117 (1984).

10. M. N. Kogan, *Rarefied Gas Dynamics* (Plenum, New York, 1969).
11. M. Lachowicz, *Bull. Pol. Acad. Sci.* **1–2**:89 (1983).
12. M. Lachowicz, *Math. Meth. Appl. Sci.* **3**:342 (1987).
13. P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).
14. G. Toscani, *Arch. Rat. Mech. Anal.* **1**:37 (1986).
15. G. Toscani and N. Bellomo, in Advances in Hyperbolic Equations, M. Witten, ed., *Comp. Math. Appl.* **3–4** (1987).